

# CLASSIFICATION OF HOMOGENEOUS FOURIER MATRICES ASSOCIATED WITH MODULAR DATA

GURMAIL SINGH

**ABSTRACT.** Modular data are commonly studied in mathematics and physics, see [10] and [12]. A modular datum defines finite dimensional representations of the modular group  $\mathrm{SL}_2(\mathbb{Z})$ . For every Fourier matrix in a modular datum there exists a self-dual  $C$ -algebra. A Fourier matrix associated with a homogeneous  $C$ -algebra is called a homogeneous Fourier matrix, that is, a Fourier matrix with all equal entries of its first row except the first entry. In this paper we classify the homogeneous Fourier matrices associated with modular data and quasi-modular data. After dividing each row of a Fourier matrix with the first entry of the row we get a new matrix that we call an Allen matrix. We prove that there is a one-to-one correspondence between Allen matrices and self-dual  $C$ -algebras that satisfy the Allen integrality condition. Also, we establish some results to find the torsion diagonal matrices associated with a homogeneous Fourier matrix in a modular datum.

## 1. INTRODUCTION

A *reality based algebra* (RBA) is a pair  $(A, \mathbf{B})$ , where  $A$  is a finite-dimensional algebra over  $\mathbb{C}$  with an  $\mathbb{R}$ -linear and  $\mathbb{C}$ -conjugate linear involution  $*$  :  $A \rightarrow A$ , and  $\mathbf{B} = \{b_0, b_1, \dots, b_{r-1}\}$  is a distinguished basis of  $A$  that satisfies the following properties:

- (i)  $b_0 = b_0^* = 1_A \in \mathbf{B}$ ;
- (ii) there is a transposition  $*$  of  $\{1, \dots, r-1\}$  such that  $(b_i)^* = b_{i^*}$ , for all  $i \in \{1, \dots, r-1\}$ ,
- (iii) the structure constants of  $A$  with respect to the basis  $\mathbf{B}$  are real numbers, i.e. for all  $b_i, b_j \in \mathbf{B}$ , we have

$$b_i b_j = \sum_{b_k \in \mathbf{B}} \lambda_{ijk} b_k, \text{ for some } \lambda_{ijk} \in \mathbb{R},$$

- (iv) for all  $b_i, b_j \in \mathbf{B}$ ,  $\lambda_{ij0} \neq 0 \iff j = i^*$ ,
- (v) for all  $b_i \in \mathbf{B}$ ,  $\lambda_{ii^*0} = \lambda_{i^*i0} > 0$ .

If these conditions are satisfied, we say that  $\mathbf{B}$  is an RBA-basis. The algebra  $A$  is a  $C^*$ -algebra whose involution satisfies  $\alpha^* = \sum_i \bar{\alpha} b_{i^*}$ , for all  $\alpha = \sum_i \alpha_i b_i \in A$ . In particular,  $A$  is a  $r$ -dimensional semisimple algebra (see [13]).

Reality-based algebras were introduced by Blau in [7] to unify treatments of various types of hypergroups. Group algebras of finite groups, Bose-Mesner algebras of finite association schemes, table algebras,  $C$ -algebras, and fusion rule algebras are all special types of reality-based algebras. Many of their basic properties were established earlier (see [1] and [16]).

An RBA  $(A, \mathbf{B})$  has a *positive degree map* if there is an algebra homomorphism  $\delta : A \rightarrow \mathbb{C}$  for which  $\delta(b_i) = \delta(b_{i^*}) > 0$  for all  $b_i \in \mathbf{B}$ . If  $(A, \mathbf{B})$  is an RBA with positive degree map  $\delta$ , then we say that its RBA-basis  $\mathbf{B}$  is *standard* when  $\delta(b_i) = \lambda_{ii^*0}$ , for all  $b_i \in \mathbf{B}$ . For each  $i \neq 0$ ,  $\delta(b_i)$  is called a nontrivial degree and  $\delta(b_0) = 1$  is called a trivial degree. Every RBA with positive degree

---

2000 *Mathematics Subject Classification.* Primary 05E30, Secondary , 05E99, 81R05.

*Key words and phrases.* Integral modular data, Fusion rings,  $C$ -algebra, Reality-based algebra.

map has a unique standard RBA-basis, which can be arranged by rescaling each basis element by  $\delta(b_i)/\lambda_{ii^*0}$ . A commutative RBA with a positive degree map is called a *C-algebra*. An RBA with a positive degree map and nonnegative structure constants is called a *table algebra*.

The set of *linear elements* of a standardized RBA basis  $\mathbf{B}$  is  $L(\mathbf{B}) = \{b_i \in \mathbf{B} : b_i b_i^* = \lambda_{ii^*0} b_0\}$ . Note that  $b_0 b_0^* = b_0^2 = b_0$ , so  $b_0 \in L(\mathbf{B})$ . Furthermore, if  $b_i \in \mathbf{B}$  then applying the degree map to  $b_i b_i^* = \lambda_{ii^*0} b_0$  gives  $\delta(b_i)^2 = \delta(b_i)$ . Since  $\delta(b_i) > 0$  we must have  $\lambda_{ii^*0} = \delta(b_i) = 1$ , and so  $b_i^* = (b_i)^{-1}$ . Therefore,  $L(\mathbf{B})$  is a finite group.

When the RBA  $(A, \mathbf{B})$  admits a positive degree map  $\delta$ , we define its *order* to be

$$n := \delta(\mathbf{B}^+) = \sum_{i=0}^d \delta(b_i),$$

and its *standard feasible trace* to be  $\rho : A \rightarrow \mathbb{C}$  with  $\rho(\alpha) = n\alpha_0$  for all  $\alpha = \sum_i \alpha_i b_i \in A$ . Note that  $\rho$  really is a trace function on  $A$  because  $\rho(\alpha\beta) = \rho(\beta\alpha)$  for all  $\alpha, \beta \in A$ . Positivity of the degree map makes  $A$  into a Frobenius algebra with nondegenerate hermitian form

$$\langle \alpha, \beta \rangle = \rho(\alpha\beta^*), \quad \text{for all } \alpha, \beta \in A.$$

Being a nonsingular trace function on the finite-dimensional Frobenius algebra  $A$ ,  $\rho$  can be expressed as a linear combination of the irreducible characters of  $A$  [16]. Let  $\text{Irr}(A)$  denote the set of irreducible characters of  $A$ . The coefficients  $m_\chi$  in this linear combination  $\rho = \sum_\chi m_\chi \chi$  are the *multiplicities* of an RBA with positive degree map, where sum runs over  $\chi \in \text{Irr}(A)$ . The multiplicities are always positive real numbers, see [7]. The standard feasible trace is a character, known as *standard character* precisely when all the multiplicities are positive rational integers, see [3], [7] and [15]. The standard feasible trace is a *pseudo-standard character* if all the multiplicities are not integers but rational numbers, see [18].

Let  $(A, \mathbf{B})$  be an RBA that has a positive degree map  $\delta$  and order  $n$ . Higman's character formula

$$e_\chi = \frac{m_\chi}{n} \sum_i \frac{\chi(b_i^*)}{\lambda_{ii^*0}} b_i$$

expresses the centrally primitive idempotents of  $A$  in terms of the standardized basis  $\mathbf{B}$ , see [16]. Since  $\chi(b_i^*) = \overline{\chi(b_i)}$  for all  $b_i \in \mathbf{B}$ , we have

$$\chi(e_\chi) = \chi(b_0) = \frac{m_\chi}{n} \sum_i \frac{\overline{\chi(b_i)} \chi(b_i)}{\lambda_{ii^*0}}.$$

$\chi(b_0)$  is the degree of  $\chi$  so it is always a positive integer. When the degrees  $\delta(b_i)$  are all positive and real, then  $n$  is positive and real, so the multiplicity  $m_\chi$  is positive and real (see [6]). It follows then that  $e_\chi^* = e_\chi$ . From the above expression for central idempotents  $e_\chi$  we obtain the orthogonality relation [16]

$$\sum_{k=0}^d \frac{\chi_i(b_k^*)}{\lambda_{kk^*0}} \chi_j(b_k) = \delta_{ij} \frac{\chi_i(b_0) \delta(\mathbf{B}^+)}{m_{\chi_i}}.$$

In Section 2, we construct the RBAs with a basis consists of the columns of Allen matrices, the matrices derived from Fourier matrices, such that the componentwise multiplication on the columns generate integral structure constants. In Section 3, we prove that every RBA arising from an Allen matrix is in fact a *C-algebra*. Then we establish some properties of *C-algebras* that arise from Allen matrices. Also, we prove that there is a one-to-one correspondence between Allen matrices and the self-dual *C-algebras* that satisfy the Allen integrality condition. That is, a *C-algebra* arises from

an Allen matrix if and only if it is self-dual and satisfies the Allen integrality condition. A Fourier matrix associated with a homogeneous  $C$ -algebra is called a homogeneous Fourier matrix, that is, a Fourier matrix with all equal entries of its first row except the first entry. In Section 4, we prove that every  $C$ -algebra that is either homogenous or of prime order has all degrees equal to 1. Then we classify the homogenous Allen matrices and the homogenous Fourier matrices under a condition that does not require all structure constants to be nonnegative. A matrix with finite multiplicative order is called a torsion matrix. In Section 5, we establish some results to find the torsion diagonal matrices associated with Fourier matrices in a modular datum from another given modular datum.

## 2. CONSTRUCTION OF RBAS ARISING FROM ALLEN MATRICES

In this section, we give definitions of the Fourier matrices and construct RBAs from them. The Fourier matrices are commonly studied in investigations of modular data and fusion algebras arising from them, see [9] and [10]. The special cases include Kac-Peterson matrices, Hadamard matrices, and the matrices corresponding to the Grothendieck rings of finite groups, see [9], [10] and [12]. Modular data and quasi-modular data are defined differently in the literature. To keep the generality, we assume the structure constants to be integers instead of nonnegative integers, see [5], [9], [10] and [12].

**Definition 1.** Let  $r \in \mathbb{Z}^+$  and  $I$  be an  $r \times r$  identity matrix. A pair  $(S, T)$  of  $r \times r$  complex matrices is called modular datum (quasi-modular datum, respectively) if

- (i)  $S$  is a unitary and symmetric matrix, so  $S\bar{S}^T = 1, S = S^T$ ,
- (ii)  $T$  is diagonal matrix and of finite order,
- (iii)  $S_{i0} > 0$  for  $0 \leq i \leq r-1$ ,
- (iv)  $(ST)^3 = S^2$  ( $(ST)^3 = I$ , respectively),
- (v)  $N_{ijk} = \sum_l S_{li} S_{lj} \bar{S}_{lk} S_{l0}^{-1} \in \mathbb{Z}$ , for all  $0 \leq i, j, k \leq r-1$ .

**Definition 2.** A matrix  $S$  satisfying the axioms (i), (iii) and (v) of Definition 1 is called a Fourier matrix.

**Definition 3.** Let  $S$  be a Fourier matrix. We call a matrix  $s = [s_{ij}]$  an Allen matrix if  $s_{ij} = S_{ij} S_{i0}^{-1}$  for all  $i, j$ .

The entries of an Allen matrix are algebraic integers, see [10] and [7, Proposition 2.17]. Therefore, in fact, an Allen matrix with rational entries has rational integer entries. Since  $s_{ij} = S_{ij} S_{i0}^{-1}$  for all  $i, j$ ,  $S \in \mathbb{R}^{r \times r}$  if and only if  $s \in \mathbb{R}^{r \times r}$ . Thus we have the following definition.

**Definition 4.** Let  $S$  be a Fourier matrix and  $s$  be the corresponding Allen matrix. If  $S \in \mathbb{R}^{r \times r}$  ( $S \in \mathbb{Q}^{r \times r}$ , resp.) then we call the corresponding Allen matrix  $s$  a real (integral, resp.) Allen matrix.

Note that for an Allen matrix  $s$ ,  $s_{i0} = 1$  for all  $0 \leq i \leq r-1$  and  $s\bar{s}^T$  is a diagonal matrix. If  $s\bar{s}^T = \text{diag}(d_0, d_1, \dots, d_{r-1})$ , where  $d_i = \sum_j s_{ij} \bar{s}_{ij}$  then  $N_{ijk} = \sum_l s_{li} s_{lj} \bar{s}_{lk} d_l^{-1}$  for all  $0 \leq i, j, k \leq r-1$ . Also,  $d_i |s_{ji}|^2 = d_j |s_{ij}|^2$ , for all  $0 \leq i, j \leq r-1$ . If an Allen matrix  $s$  and an associated diagonal matrix  $T$  are integral matrices then the modular datum  $(s, T)$  is called an *integral modular datum* and the Allen matrix  $s$  is called an *integral Fourier matrix*. The study of integral modular datum is of independent interest. The following definitions of integral modular datum and integral Fourier matrix are from Cuntz's paper [10].

**Definition 5.** Let  $r \in \mathbb{Z}^+$ . A pair  $(s, T)$  of  $r \times r$  integral matrices is called an *integral modular datum* if

- (i)  $s_{i0} = 1$  for  $0 \leq i \leq r-1$ ,  $\det(s) \neq 0$ ,
- (ii)  $ss^T = \text{diag}(d_0, d_1, \dots, d_d)$ , where  $d_i = \sum_j s_{ij}^2$ ,
- (iii)  $\sqrt{d_j}s_{ij} = \sqrt{d_i}s_{ji}$  for  $0 \leq i, j \leq r-1$ , so  $s$  is symmetrizable,
- (iv)  $N_{ijk} = \sum_l s_{li}s_{lj}s_{lk}d_l^{-1} \in \mathbb{Z}$ , for all  $0 \leq i, j, k \leq r-1$ ,
- (v)  $T$  is diagonal,
- (vi)  $S^2 = (ST)^3$ , where  $S_{ij} = s_{ij}/\sqrt{d_i}$ .

**Definition 6.** A matrix  $s$  satisfying the axioms (i), (iii) and (v) of Definition 5 is called an integral Fourier matrix.

Thus an integral Fourier matrix is in fact an integral Allen matrix. Therefore, every result that is true for Allen matrices is also true for integral Fourier matrices. Also, we will see that the real Allen matrices of a specific class become integral Fourier matrices, see Theorem 25. But an Allen matrix need not be an integral Fourier matrix, for example, the character table of cyclic group of order 4 is an Allen matrix but not an integral Fourier matrix.

There is an interesting row-and-column operation procedure that can be applied to the first eigenmatrix  $P$ , the character table, of a self-dual  $C$ -algebra that results in a unitary matrix  $S$ . This unitary matrix  $S$  will be symmetric when it is the Fourier matrix occurring in a modular datum (see [10] and [12]). Reversing the procedure starting with  $\bar{S}^T$  results in the second eigenmatrix  $Q$ . (Even when  $P$  is the character table of a non-commutative RBA with  $h$  irreducible characters, this process results in an  $h \times r$ -matrix  $S$  for which  $S\bar{S}^T = I$ , an identity matrix.) This procedure is more illuminating than simply calculating the inverse of  $P$  to find the multiplicities. But there is an intermediate matrix  $s$  between the matrices  $P$  and  $S$ . We divide each column of  $P$  by  $\sqrt{P_{0j}}$ . We call the result an Allen matrix and denote it by  $s$ . For each  $i$ , we divide the  $i$ th row of  $s$  by  $\sqrt{d_i}$ , where  $d_i = \sum_j |s_{ij}|^2$  and we call the result  $S$ . We will illustrate the procedure with an example from [10] that arises in constructions of modular data.

**Example 7.** We will illustrate the procedure with the character table of the association scheme as9-5 (though we remark the procedure applies to the first eigenmatrix of any commutative RBA):

$$P = \begin{bmatrix} 1 & 4 & 4 \\ 1 & 1 & -2 \\ 1 & -2 & 1 \end{bmatrix}.$$

We label the rows and columns with  $0, 1, 2, \dots$ , and by convention the first row will correspond to the degree map  $\delta$  and the first column to  $b_0$ .

**Step 1.** Divide each column of  $P$  by  $\sqrt{P_{0j}}$ . We call the result  $s$ . For our example we get:

$$s = \begin{bmatrix} 1 & 2 & 2 \\ 1 & \frac{1}{2} & -1 \\ 1 & -1 & \frac{1}{2} \end{bmatrix}.$$

To reverse Step 1, we multiply the  $j$ -th column by  $s_{0j}$  to get the matrix  $P$  from an Allen matrix  $s$ .

**Step 2.** Divide each row of  $s$  by  $\sqrt{d_i}$ , where  $d_i = \sum_j |s_{ij}|^2$ . Note that with our labelling conventions,  $d_0 = n$  and by Proposition 16,  $d_i = \frac{d_0}{m_i}$ , where  $m_i$  is the multiplicity of the  $i$ -th irreducible character. By orthogonality relations for characters, the resulting matrix  $S$  is a unitary matrix. For our example,  $S$  is symmetric, because we started with the eigenmatrix of a self-dual

association scheme.

$$S = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}.$$

To reverse Step 2, divide the  $i$ -th row by its first entry  $S_{i0}$  to get the Allen matrix from a Fourier matrix  $S$ .

To obtain  $Q$ , start with  $\bar{S}^\top$ , apply the reverse of Step 2 then the reverse of Step 1.

**Remark 8.** Throughout this paper, unless mentioned explicitly,  $S(=[S_{ij}])$  denotes a Fourier matrix. We use the notation  $s(=[s_{ij}])$  and  $P(=[P_{ij}])$  for the Allen matrix and the first eigenmatrix, respectively. We label the rows and columns with  $0, 1, 2, \dots$ . The set of the columns of matrices  $P$  and  $s$  are denoted by  $\mathbf{B} = \{b_0, b_1, \dots, b_{r-1}\}$  and  $\tilde{\mathbf{B}} = \{\tilde{b}_0, \tilde{b}_1, \dots, \tilde{b}_{r-1}\}$ , respectively. The structure constants generated by the columns, with the componentwise multiplication, of  $P$  and  $s$  are denoted by  $\lambda_{ijk}$  and  $N_{ijk}$ , respectively. For any matrix  $B$  the transpose of  $B$  is denoted by  $B^T$ . A primitive  $n$ th root of unity is denoted by  $\zeta_n$ .

In the next lemma we show that corresponding to every Allen matrix there is an RBA.

**Lemma 9.** *Let  $S$  be a Fourier matrix. Let  $\mathbf{B}$  and  $\tilde{\mathbf{B}}$  be the sets of the columns of matrices  $P$  and  $s$ , respectively. Then the vector spaces  $\mathbb{C}\mathbf{B}$  and  $\mathbb{C}\tilde{\mathbf{B}}$  over the complex field are RBAs.*

*Proof.* The  $\mathbb{C}$ -conjugate linear involution  $*$  on columns of  $S$  is given by the involution on elements of  $S$ , defined as  $(S_{ij})^* = S_{ij^*} = \bar{S}_{ij}$  for all  $i, j$ . If  $S_j$  denotes the  $j$ th column of a Fourier matrix  $S$  then the involution on  $S_j$  is defined as  $(S_j)^* = S_{j^*} = [S_{0j^*}, S_{1j^*}, \dots, S_{(r-1)j^*}]^T$ . Since  $b_i = s_{0i}\tilde{b}_i$ , the structure constants generated by the basis  $\mathbf{B}$  are given by  $\lambda_{ijk} = N_{ijk}s_{0i}s_{0j}s_{0k}^{-1}$ , for all  $i, j, k$ . As  $S$  is a unitary and symmetric matrix, therefore,  $N_{ij0} = \sum_l S_{li}S_{lj}\bar{S}_{l0}S_{l0}^{-1} \neq 0 \iff j = i^*$  and  $N_{ii^*0} = 1 > 0$  for all  $i, j$ . Hence,  $\lambda_{ij0} \neq 0 \iff j = i^*$  and  $\lambda_{ii^*0} > 0$  for all  $i, j$ . Also, the first column each of the matrices  $P$  and  $s$  is an identity element. Therefore, the  $\mathbb{C}$ -span of the elements of  $\mathbf{B}$  and  $\tilde{\mathbf{B}}$  are RBAs.  $\square$

**Definition 10.** *Let  $\mathbf{B} = \{b_0, b_1, \dots, b_{r-1}\}$  denote the columns of a matrix  $P$  obtained from an Allen matrix (integral Fourier matrix, respectively)  $s$ . Then the RBA  $(A, \mathbf{B})$  is called an RBA arising from an Allen matrix (integral Fourier matrix, respectively)  $s$ .*

### 3. GENERAL RESULTS ON RBAS ARISING FROM ALLEN MATRICES

In the next theorem we prove RBAs arising from Allen matrices possess a positive degree map.

**Theorem 11.** *Let  $(A, \mathbf{B})$  be an RBA arising from an Allen matrix  $s$ . Then  $A$  has the positive degree map, and  $\mathbf{B}$  is the standard basis of  $A$ .*

*Proof.* Let  $\tilde{\mathbf{B}}$  and  $\mathbf{B}$  denote the columns of  $s$  and  $P$  respectively. Define a  $\mathbb{C}$ -conjugate linear map  $\delta : A \longrightarrow \mathbb{C}$  as  $\delta(\sum_i a_i \tilde{b}_i) = \sum_i \bar{a}_i s_{0i}$ . Thus  $\delta(b_i) = \delta(s_{0i}\tilde{b}_i) = s_{0i}^2$ , hence  $\delta$  is positive valued. Since  $b_{i^*} = \bar{s}_{0i}\tilde{b}_{i^*} = s_{0i}\tilde{b}_{i^*}$ ,  $(b_i b_{i^*})_0 = s_{0i}s_{0i}(\tilde{b}_i \tilde{b}_{i^*})_0 = s_{0i}^2 = \delta(b_i)$ . The map  $\delta$  is an algebra homomorphism, to see:

$$\begin{aligned} \delta(b_i b_j) &= s_{0i}s_{0j} \sum_k N_{ijk} \delta(\tilde{b}_k) \\ &= s_{0i}s_{0j} \left[ \frac{s_{0i}s_{0j}}{d_0} \sum_{k=0}^{r-1} \bar{s}_{0k}s_{0k} \right] + s_{0i}s_{0j} \left[ \sum_{l=1}^{r-1} \frac{s_{li}s_{lj}}{d_l} \sum_{k=0}^{r-1} \bar{s}_{lk}s_{0k} \right] \\ &= s_{0i}s_{0j} \left[ \frac{s_{0i}s_{0j}}{d_0} d_0 \right] + s_{0i}s_{0j} [0] \\ &= s_{0i}^2 s_{0j}^2 = \delta(b_i) \delta(b_j) \end{aligned}$$

In the third last equality we use the fact that  $\sum_{k=1}^n s_{0k} \bar{s}_{0k} = d_0$  and the rows of  $s$  are orthogonal.  $\square$

Since an RBA arising from an Allen matrix (integral Fourier matrix) is commutative and posses a positive degree map, on forth we call it a  $C$ -algebra arising from an Allen matrix (integral Fourier matrix, respectively).

**Lemma 12.** *Let  $(A, \mathbf{B})$  be a  $C$ -algebra arising from an integral Fourier matrix  $s$  with standard basis  $\mathbf{B} = \{b_0, b_1, \dots, b_{r-1}\}$ . If  $\delta(b_i) = k_i$  for all  $i$  then  $k_i$  are square integers.*

*Proof.* Since  $\delta(b_i) = k_i$  for all  $i$ , the first row of  $s$  is  $[1, \sqrt{k_1}, \sqrt{k_2}, \dots, \sqrt{k_{r-1}}]$ . But  $s$  is an integral matrix. Therefore,  $\sqrt{k_i} \in \mathbb{Z}$  for all  $i$ .  $\square$

**Definition 13.** *Let  $(A, \mathbf{B})$  be a  $C$ -algebra arising from an Allen matrix  $s$  with standard basis  $\mathbf{B} = \{b_0, b_1, \dots, b_{r-1}\}$ . Let  $k \in \mathbb{Z}^+$  and  $\delta(b_i) = k$  for all  $1 \leq i \leq r-1$ . Then  $A$  is called a homogenous  $C$ -algebra with homogeneity degree  $k$ , and the associated Fourier matrix  $S$  (Allen matrix  $s$ , respectively) is called a homogeneous Fourier matrix (homogeneous Allen matrix, respectively).*

**Definition 14.** *Let  $(A, \mathbf{B})$  be a  $C$ -algebra arising from an Allen matrix (integral Fourier matrix)  $s$ . The magnitudes of the row vectors of  $s$ ,  $d_0, d_1, \dots, d_{r-1}$  are called norms. We call  $d_0$  the principal norm and  $d_1, d_2, \dots, d_{r-1}$  non-principal norms.*

The principal norm  $d_0$  is also known as the size of a modular datum. In fact, the principal norm  $d_0$  is the order of a  $C$ -algebra arising from an Allen matrix (integral Fourier matrix)  $s$ . A  $C$ -algebra arising from an Allen matrix (integral Fourier matrix)  $s$  have rational multiplicities by their inherent self-duality property, that comes from the symmetry of the Fourier matrix  $S$ . In the next lemma, we prove that the multiplicities (degrees) and norms are in fact rational integers.

**Proposition 15.** *Let  $(A, \mathbf{B})$  be a  $C$ -algebra arising from an Allen matrix  $s$ . Then the norms and the degrees of  $A$  are rational integers.*

*Proof.* Since the entries of  $s$  are algebraic integers, the matrix  $P$  has algebraic integer entries, see [7, Proposition 2.17], [10]. Therefore, magnitude of the row vectors of  $s$  are rational integers. The degrees (multiplicities) of  $A$  are rational numbers because  $A$  is self dual. Hence they are rational integers.  $\square$

In the next proposition, we prove that the degrees (multiplicities) of a  $C$ -algebra arising from an Allen matrix divide the order of the  $C$ -algebra. In addition, we prove that not only the list of multiplicities matches with the list of degrees but also their indices match.

**Proposition 16.** *Let  $(A, \mathbf{B})$  be a  $C$ -algebra arising from Allen matrix  $s$ .*

- (i) *The multiplicities (degrees) divide the order of  $A$ , Also,  $m_j = d_0/d_j$  for all  $j$ .*
- (ii) *The degrees of  $A$  exactly match with the multiplicities of  $A$ , that is,  $m_j = \delta(b_j)$  for all  $j$ .*
- (iii) *If the algebra  $A$  is homogenous or order of  $A$  is prime, then the column norm is equal to the corresponding row norm, that is,  $\sum_j s_{ij} \bar{s}_{ij} = \sum_j s_{ji} \bar{s}_{ji}$ .*

*Proof.* For (i). Let  $m_j$  denote the multiplicity corresponding to the irreducible character  $\chi_j$ . Now, for  $0 \leq j \leq r-1$ , we have

$$m_j = \delta(\mathbf{B}^+) \chi_j(b_0) / \sum_{i=0}^{r-1} \frac{|\chi_j(b_i^*)|^2}{\lambda_{ii^*0}} = d_0 / \sum_{i=0}^{r-1} \frac{|s_{ji} s_{0i}|^2}{\delta(b_i)} = d_0 / \sum_{i=0}^{r-1} |s_{ji}|^2 = d_0/d_j.$$

For (ii). By Part (i),  $m_j = d_0/d_j$  for all  $j$ . Also,  $d_i|s_{ji}|^2 = d_j|s_{ij}|^2$  for all  $0 \leq i, j \leq r-1$ . Therefore,  $d_0/d_j = |s_{0j}|^2 = s_{0j}^2$ . Hence  $m_j = s_{0j}^2 = \delta(b_j)$  for all  $j$ .

For (iii). By Theorem 23,  $d_j = d_0$  for all  $j$ . Since  $d_i^{-1} = S_{i0}\bar{S}_{i0}$ , we have

$$\sum_j s_{ji}\bar{s}_{ji} = \sum_j s_{ij}\bar{s}_{ij} \frac{d_j}{d_i} = \sum_j S_{ij}\bar{S}_{ij}d_j = \sum_j S_{ji}\bar{S}_{ji}d_j = \sum_j s_{ij}\bar{s}_{ij}$$

□

Let  $(A, \mathbf{B})$  be a  $C$ -algebra arising from an Allen matrix (integral Fourier matrix)  $s$ . Then  $A$  has positive integral multiplicities, see Proposition 15 and Proposition 16. Thus  $A$  has the standard character, see [3], [15] and [18].

**Definition 17.** Let  $(A, \mathbf{B})$  be a  $C$ -algebra with standard basis  $\mathbf{B} = \{b_0, b_1, \dots, b_{r-1}\}$ . We say that  $A$  satisfy the Allen integrality condition if  $\lambda_{ijk}\sqrt{\delta(b_k)}/\sqrt{\delta(b_i)\delta(b_j)} \in \mathbb{Z}$  for all  $i, j, k$ .

Note that every group algebra  $\mathbb{C}G$  for an abelian group  $G$  satisfy the Allen integrality condition. If a  $C$ -algebra is arising from an Allen matrix then it is self-dual. But every self-dual  $C$ -algebra not necessarily arise from an Allen matrix, see Example 20. Therefore, in general, the converse is not true. In the next theorem we prove that the converse is also true for the self-dual  $C$ -algebras satisfying the Allen integrality condition. After establishing the following theorem the author of this paper noted that a similar result is proved by Eiichi Bannai for fusion algebras at algebraic level with real structure constants, see [4, Theorem 3.1].

**Theorem 18.** Let  $(A, \mathbf{B})$  be a  $C$ -algebra with standard basis  $\mathbf{B} = \{b_0, b_1, \dots, b_{r-1}\}$ . Then  $A$  is self-dual and satisfies the Allen integrality condition if and only if  $A$  arises from an Allen matrix.

*Proof.* Let  $(A, \mathbf{B})$  be a  $C$ -algebra arising from an Allen matrix  $s$ . Therefore,  $A$  satisfies the Allen integrality condition. Since the Fourier matrix  $S$  is symmetric and unitary,  $A$  is self-dual. Conversely, suppose  $(A, \mathbf{B})$  is a self-dual  $C$ -algebra that satisfies the Allen integrality condition. Let  $P$  be the character table of  $A$  and  $I$  be the identity matrix. As complex conjugation is realized by a column permutation, therefore, without loss of generality, let  $P\bar{P} = d_0I$ . Otherwise, we can perform suitable row/column permutations on  $P$ , see [18]. Let  $L = \text{diag}(1/\sqrt{\delta(b_0)}, 1/\sqrt{\delta(b_1)}, \dots, 1/\sqrt{\delta(b_{r-1})})$ , and  $P_1 = PL$ . Therefore,  $P_1\bar{P}_1^T = \text{diag}(d_0/\delta(b_0), d_0/\delta(b_1), \dots, d_0/\delta(b_{r-1})) = \text{diag}(d_0, d_1, \dots, d_{r-1})$ , see Proposition 16. The matrix  $P_1$  is an Allen matrix. □

**Remark 19.** By the above theorem, in particular, every self-dual integral  $C$ -algebra with unique degree, each degree equal to 1, is arising from an Allen matrix. If a self-dual  $C$ -algebra with unique degree has nonnegative structure constants then it is a group algebra with basis a finite abelian group, see Corollary 26. By Pontryagin duality, every group algebra for a finite abelian group is self-dual. Since every group algebra has unique degree, it satisfies the Allen integrality condition thus the character table of the group is an Allen matrix. Hence a self-dual  $C$ -algebra with nonnegative structure constants has a unique degree if and only if it is a group algebra for a finite abelian group.

In the next example, we show that the Allen integrality condition cannot be removed from the above theorem.

**Example 20.** Let  $n > 1$  be a positive integer. Then  $P = \begin{bmatrix} 1 & n \\ 1 & -1 \end{bmatrix}$  is the character table of the association scheme of order  $n+1$  and rank 2. The corresponding  $s = \begin{bmatrix} 1 & \sqrt{n} \\ 1 & -1/\sqrt{n} \end{bmatrix}$

and  $S = \frac{1}{\sqrt{n+1}} \begin{bmatrix} 1 & \sqrt{n} \\ \sqrt{n} & -1 \end{bmatrix}$ . The structure constants  $\lambda_{ijk}$ , generated by the columns of the matrix  $P$ , are integers. But the structure constant generated by the columns of the matrix  $S$ ,  $N_{111} = (n-1)/\sqrt{n}$ , is not an integer for  $n > 1$ . Therefore, the Allen integrality condition is not satisfied and  $s$  is not an Allen matrix. Hence, for  $n > 1$ , the adjacency algebra of rank 2 and order  $n+1$  is not arising from an Allen matrix.

Let  $(A, \mathbf{B})$  be a  $C$ -algebra with standard basis  $\mathbf{B} = \{b_0, b_1, \dots, b_{r-1}\}$ . Then  $A$  is called *symmetric* if  $b_{i*} = b_i$  for all  $i$ . In the next proposition, we prove that a  $C$ -algebra arising from an Allen matrix is a symmetric  $C$ -algebra if and only if the Allen matrix is a real matrix. Note that the integral entries of the first eigenmatrix  $P$  does not imply that the algebra is arising from an integral Fourier matrix.

**Proposition 21.** *Let  $(A, \mathbf{B})$  be a  $C$ -algebra arising from an Allen matrix  $s$  with rank  $r$ .*

- (i) *Then  $A$  is a symmetric  $C$ -algebra if and only if  $s \in \mathbb{R}^{r \times r}$ .*
- (ii) *If the entries of matrix  $P$  are integers then the order of  $A$  is a square integer.*

*Proof.* For (i). Let  $\bar{S}_{li} = S_{li}$ , for all  $0 \leq i, l \leq r-1$ . Therefore,  $N_{ii0} = \sum_l S_{l0}^{-1} S_{li} S_{li} \bar{S}_{l0} = \sum_l S_{li} \bar{S}_{li} = 1 \neq 0$ . Thus  $b_{i*} = b_i$  for all  $i$ . Conversely, let  $A$  be a symmetric  $C$ -algebra. Therefore,  $b_{i*} = b_i$  for all  $i$ . As  $b_{i*} = \bar{s}_{0i} \tilde{b}_{i*}$ , therefore  $\tilde{b}_{i*} = \tilde{b}_i$ , for all  $i$ . Hence the columns of  $s$  are real.

For (ii). Let  $n(= d_0)$  be the order of  $A$ . Since  $A$  is self dual,  $P\bar{P} = nI$ , thus  $(\det(P))^2 = n$ . Hence  $\sqrt{n}$  is an integer.  $\square$

#### 4. HOMOGENEOUS $C$ -ALGEBRAS ARISING FROM ALLEN MATRICES

In the next theorem we prove that if a degree of a  $C$ -algebra arising from an Allen matrix divides the all other non-trivial degrees then that degree might be equal to 1.

**Theorem 22.** *Let  $(A, \mathbf{B})$  be a  $C$ -algebra arising from an Allen matrix with standard basis  $\mathbf{B} = \{b_0, b_1, \dots, b_{r-1}\}$ . Let  $\delta(b_i) = k_i$  for all  $i \geq 1$ . If for a given  $j$ ,  $k_j$  divides  $k_i$  for all  $i \geq 1$  then  $k_j = 1$ .*

*Proof.* By Proposition 16,  $d_0 k_j^{-1} = d_j$  for all  $j$ . Since  $d_0 = 1 + \sum_{i=1}^d k_i$  and  $d_j \in \mathbb{Z}$ ,  $(1 + \sum_{i=1}^d k_i) k_j^{-1} = k_j^{-1} + \alpha \in \mathbb{Z}$ , where  $\alpha \in \mathbb{Z}$ . Hence  $k_j = 1$ .  $\square$

Note that, the above theorem is true for any self-dual  $C$ -algebra with integral norms and degrees. The theorem has very useful applications. We can apply above theorem to recognize some  $C$ -algebras not arising from Allen matrices (or integral Fourier matrices) just by looking at the degree pattern (or first row of the character table) of a  $C$ -algebra. For example, the character tables of the association schemes **as5(2)**, **as9(2)**, **as9(3)**, **as9(8)** and **as9(9)** violates the conditions of the above theorem so they are not character tables of the adjacency algebras arising from Allen matrices (or integral Fourier matrices), for the character tables see [2].

In the next theorem, we prove that if a  $C$ -algebra is arising from an Allen matrix is either homogeneous or of prime order then each degree of the algebra is equal to 1.

**Theorem 23.** *Let  $(A, \mathbf{B})$  be a  $C$ -algebra arising from an Allen matrix  $s$  with standard basis  $\mathbf{B} = \{b_0, b_1, \dots, b_{r-1}\}$ .*

- (i) *If  $A$  is homogenous then  $\delta(b_i) = 1$  for all  $i$ .*



(ii) *If the order of  $A$  is a prime number then  $\delta(b_i) = 1$  for all  $i$ .*

*Proof.* For (i). Since  $\delta(b_1) = \delta(b_i)$  for all  $i > 0$ ,  $\delta(b_1)$  divides  $\delta(b_i)$  for all  $i > 0$ . By Theorem 22,  $\delta(b_1) = 1$ , hence  $\delta(b_i) = 1$  for all  $i$ .

For (ii). The order of  $A$  is  $d_0 (= \delta(\mathbf{B}^+))$ . By Proposition 16,  $\delta(b_i)$  divides  $d_0$  for all  $i$ . But  $d_0$  is prime, thus  $\delta(b_i) = 1$  for all  $i$ .  $\square$

By unique norm, we mean  $d_i = d_0$  for all  $i$ . The following proposition summarizes the equivalent conditions when a  $C$ -algebra arising from an Allen matrix has a unique norm.

**Proposition 24.** *Let  $(A, \mathbf{B})$  be a  $C$ -algebra arising from an Allen matrix  $s$  with standard basis  $\mathbf{B} = \{b_0, b_1, \dots, b_{r-1}\}$ . Then the following conditions are equivalent.*

- (i)  *$A$  has a unique norm.*
- (ii)  *$A$  has unique degree, that is,  $\delta(b_i) = 1$  for all  $i$ .*
- (iii)  *$A$  is homogenous, and for the character table of  $A$  the column sum is zero for the all columns but the first column.*

*Proof.* (i)  $\implies$  (ii). Let  $d_i = d_0$  for all  $i$ . By Proposition 16,  $\delta(b_i) = d_0 d_i^{-1} = 1$  for all  $i$ .

(ii)  $\implies$  (iii). Let  $P$  be the character table of  $A$  and  $\delta = \chi_1, \chi_2, \dots, \chi_r$  be the irreducible characters of  $A$ . Since  $S$  is symmetric and  $\delta(b_i) = 1$  for all  $i$ , the matrix  $P$  is symmetric. The assertion follows from the fact that the row sum of the character table is zero.

(iii)  $\implies$  (i). Let  $\delta(b_i) = k_i$  for all  $i > 0$ . Therefore, the  $j$ th column of the matrix  $P$  is  $[k_j, \chi_j(b_1), \chi_j(b_2), \dots, \chi_j(b_{r-1})]$ . Since the row and column sum is zero,  $k_i = 1$ . Therefore, by Proposition 16,  $d_j = d_0$  for all  $j$ .  $\square$

Michael Cuntz made a conjecture that is a generalization to his result, see [10, Lemma 3.11]. The conjecture states that if  $s$  is an integral Fourier matrix with unique norm and  $|s_{ij}| \leq s_{0j}$  for all  $j$ , then  $s \in \{\pm 1\}^{r \times r}$ , see [10, Conjecture 3.10]. Let's consider the character table of a cyclic group of order 4. Since each degree is equal to 1, the group algebra has unique norm, see Proposition 24. But the character table does not have only  $\pm 1$  entries. Thus the matrix  $s$  cannot be a non-real matrix. By Proposition 16,  $d_0 \delta(b_i)^{-1} = d_i$ , thus  $d_1 = \dots = d_{r-1}$  if and only if  $\delta(b_1) = \dots = \delta(b_{r-1})$ . The next theorem completely classify the Allen matrices under the condition that does not require all structure constants to be nonnegative. It also shows that even for real Allen matrices it is not necessary to equate all the non-principal norms to the principal norm to get all entries of Allen matrices equal to  $\pm 1$ .

**Theorem 25.** *Let  $(A, \mathbf{B})$  be a  $C$ -algebra arising from an Allen matrix  $s$  such that  $A$  is either homogeneous or order of  $A$  is prime. Let  $|s_{ij}| \leq s_{0j}$  for all  $i, j$ .*

- (i) *Then  $|s_{ij}| = 1$  for all  $i, j$ .*
- (ii) *If  $s$  is a non-real Allen matrix then  $s$  is the character table of either cyclic group or non-elementary abelian group of order  $r$ .*
- (iii) *If  $s$  is a real Allen matrix then  $s$  is the character table of an elementary abelian group of order  $r$ .*

*Proof.* For (i). Since  $A$  is homogeneous, by Theorem 23,  $\delta(b_j) = 1$  for all  $j$ . Thus  $s_{0j} = 1$  for all  $j$ . The assertion follows as  $s$  has algebraic integer entries.

For (ii). Clearly, the character table matrix  $P (= s)$  is symmetric. Let  $s_i$  be the  $i$ -th column of  $s$  and componentwise multiplication of  $s_i$  and  $s_j$  be  $s_i s_j$ . Since the multiplication of two linear characters is a linear (irreducible) character,  $s_i s_j = s_k$  for some column  $s_k$  of  $s$ . The first column of  $s$  serves as the identity of the group. Since  $s$  is non-real, all columns of  $s$  cannot be inverse of

themselves under componentwise multiplication. Hence  $s$  is the character table of a cyclic group or non-elementary abelian group of order  $r$ .

For (iii). Let  $s$  be the real matrix. Since  $|s_{ij}| = 1$ ,  $s_{ij} = \pm 1$ . Thus, using the reasoning of Part (ii) above,  $s$  is the character table of the elementary abelian group of order  $r$ .  $\square$

As we see in Section 2 every Fourier matrices  $S$  can be obtained from the associated Allen matrix  $s$  by dividing the each row of  $s$  with square roots of the norm of that row. So once we determine the Allen matrices it not hard to find the corresponding Fourier matrices. Furthermore, if an Allen matrix  $s$  of rank  $r$  has all degrees equal to 1 then the corresponding Fourier matrix  $S = r^{-1/2}s$ . Thus the above theorem also classifies the homogeneous Fourier matrices under the same condition. Note that every  $C$ -algebra with nonnegative structure constants with character table  $P$  satisfies the condition  $|p_{ij}| \leq p_{0j}$  for  $i, j$ , see [20, Proposition 4.1]. Hence the above theorem is also true for the  $C$ -algebras with nonnegative structure constants. From the above theorem we also deduce that the set of real Allen matrices and the set of integral Fourier matrices coincide when the structure constants are nonnegative. Therefore, the following corollary is a direct consequence of the above theorem.

**Corollary 26.** *Let  $(A, B)$  be a table algebra arising from an Allen matrix  $s$  with rank  $r$  such that  $A$  is either homogeneous or order of  $A$  is prime.*

- (i). *If  $s$  is a real matrix then  $s$  is the character table of an elementary abelian group of order  $r$ .*
- (ii). *If  $s$  is a non-real matrix then  $s$  is the character table of a (non-elementary) abelian group of order  $r$ .*

The following proposition states the conditions when a homogenous  $C$ -algebra arising from an Allen matrix  $s$  does not have prime order.

**Proposition 27.** *Let  $(A, B)$  be a homogeneous  $C$ -algebra arising from an Allen matrix  $s$  such that  $|s_{ij}| \leq s_{0j}$  for all  $i, j$ .*

- (i) *If  $A$  is symmetric then the rank of  $A$  is not a prime number.*
- (ii) *If  $s$  is a real matrix then the rank of  $A$  is a power of 2 and  $A$  is symmetric.*

*Proof.* For (i). Suppose  $A$  is a symmetric matrix. By Proposition 21,  $s$  is a real matrix, thus the columns of  $s$  cannot form a cyclic group. Therefore, the rank of  $s$  is not a prime.

For (ii). Since  $s$  is a real matrix, by Theorem 25, the columns of  $s$  forms an elementary abelian group under componentwise multiplication. Therefore,  $A$  is symmetric and rank of  $A$  is a power of 2.  $\square$

## 5. THE DIAGONAL MATRIX $T$ ASSOCIATED WITH HOMOGENEOUS MATRIX $s$

Michael Cuntz classify the diagonal torsion matrices associated to the character table of elementary abelian groups, that is, the homogeneous integral Fourier matrices with nonnegative structure constants, see [10, Proposition 5.1]. So he finds a  $T$  matrix corresponding to a homogenous integral Fourier matrix  $s$  with rank  $r$  that admits the nonnegative structure constants and consequently to associated Fourier matrix  $S = r^{-1/2}s$ . Bannai and Bannai classify the diagonal torsion matrices associated with the character tables of cyclic groups, see [10, Theorem 1]. Thus they find a  $T$  matrix corresponding to a non-integral homogenous Fourier matrix  $s$  with nonnegative structure constants, where  $s$  is not a tensor product of the matrices of smaller rank. In addition, they give the conditions of existence of a diagonal torsion matrix corresponding to the character table of a given group.

**Theorem 28.** *Let  $G$  be a cyclic group. Let  $\zeta$  be the  $|G|$  root of unity. The character table  $P$  of  $G$  has the modular invariance property,  $(s, T)$  is a modular datum, with a diagonal matrix  $T = \text{diag}(\alpha_0, \alpha_1, \dots, \alpha_{r-1})$  if and only if the following holds:*

$\alpha_i = \eta^{i^2} \alpha_0$  for  $i \in \{0, 1, \dots, r-1\}$  and  $\alpha_0^3 = \sqrt{r} / \sum_{i=0}^{r-1} \eta^{i^2}$ , where  $\eta = \zeta^{\frac{r-1}{2}}$  if  $r$  is odd and  $\eta^2 = \zeta^{-1}$  if  $r$  is even.

Let  $(S, T)$  be a modular datum. Since  $(S\zeta_3 T)^3 = \zeta_3^3 (ST)^3 = S^2$ ,  $(S, \zeta_3 T)$  is also a modular datum. Hence, in a modular datum, for a given Fourier matrix there might be at least three different associated torsion diagonal matrices. Therefore, for the remaining section we investigate the properties of Fourier matrices and permutation matrices establish the results that help to find the additional diagonal torsion matrices from a given diagonal torsion matrix  $T$  associated to a Fourier matrix  $S$  in a given modular datum  $(S, T)$ . In particular, the following results are useful when an Allen matrix  $s$  is not a tensor product of the Allen matrices of lower rank but can be obtained from such a matrix by permuting two or three rows (columns). In the next proposition, we observe the behavior of non-singular symmetric matrices on multiplication with permutation matrices.

**Proposition 29.** *Let  $S$  be a non-singular symmetric matrix and  $P$  be a permutation matrix. Let  $SP$  be a symmetric matrix. Then*

- (i)  $(SP)^2 = S^2$  and  $(PS)^2 = S^2$ ,
- (ii)  $PS = SP$  if and only if the order of  $P$  is 2,
- (iii) if  $S$  is a Fourier matrix and  $P_{11} = 1$  then  $SP$  and  $PS$  are Fourier matrices.

*Proof.* For (i). Since  $SP$  is a symmetric matrix,  $P^T S = SP$ , implies  $S = PSP$ . Thus  $(SP)^2 = S^2$ . Similarly  $(PS)^2 = S^2$ .

For (ii). Since  $SP$  is a symmetric matrix,  $P^T S = SP$ . Thus  $SP = PS$  if and only if  $P^T = P$ .

For (iii). Obviously, the set of the structure constants of  $PS$  is equal to the set of the structure constants of  $S$ . Also  $SP$  is a unitary matrix. Hence  $SP$  is a Fourier matrix. Similarly,  $PS$  is a Fourier matrix.  $\square$

In the next lemma, we find the conditions under which  $(SP, \tilde{T})$  becomes modular datum for a suitable diagonal matrix  $\tilde{T}$  when order of the permutation matrix  $P$  is 2.

**Lemma 30.** *Let  $(S, T)$  be a modular datum and  $P$  be a permutation matrix of order 2 that permutes the columns (rows)  $j$  and  $k$  of  $S$ , where  $1 \leq j, k \leq r-1$ . Let  $SP$  be a symmetric matrix. Let all the entries of  $T$  other than  $T_{jj}$  and  $T_{kk}$  be equal,  $T_{jj} = -T_{kk}$  and linear independent of the remaining entries. Let after the diagonalization of the matrix  $PT$  the new eigenvalues be kept at  $j$ th and  $k$ th position, and call the result  $\tilde{T}$ . Then  $(SP, \tilde{T})$  is a modular datum.*

*Proof.* Let  $R := SP$ . Therefore, for all  $0 \leq i \leq r-1$ , we have  $R_{ij} = R_{ik}$ ,  $R_{jj} = R_{kk}$ , and  $R_{jk} \neq R_{jj}$ . Note that  $\tilde{T}_{ii} = T_{ii} = x$  (say) for  $i \neq j, k$ . Let  $\tilde{T}_{jj} = z$  and  $T_{jj} = y$ . Since  $\det(\tilde{T}) = -\det(T)$ ,  $x^{r-2}z^2 = -x^{r-2}y^2$ , implies  $z^2 = -y^2$ . Also  $x^2y, xy^2$  and  $y^3$  are linear independent. Therefore,  $(S\tilde{T})^3 = (ST)^3 = (SP)^2$ , see Proposition 32.  $\square$

**Definition 31.** *Let  $S$  be a non-singular symmetric matrix and  $P_1, P_2, \dots, P_n$  be permutation matrices of order  $k$ . Let  $P_{11} = 1$  for each  $P$ . Let  $SP_1, SP_1P_2, \dots, SP_1 \dots P_n$  ( $P_1S, P_1P_2S, \dots, P_1 \dots P_nS$ ) be symmetric matrices. Then we call the set  $V := \{SP_1, SP_1P_2, \dots, SP_1 \dots P_n\}$  ( $\{P_1S, P_1P_2S, \dots, P_1 \dots P_nS\}$ , respectively) a left (right, respectively) orbit of symmetric matrices under the action of  $k$ -cycle (briefly, left (right) orbit of symmetric matrices). We call such an orbit a principal*

left (right) orbit of symmetric matrices *if it contains a symmetric matrix that is tensor product of symmetric matrices of smaller ranks and such a symmetric matrix is called the principal symmetric matrix.*

Since the character table of every abelian group can be written as a tensor product of cyclic groups of lower order, for each rank there is a principal orbit. However, it is not necessary that an orbit corresponds uniquely to order  $k$ , see Example 36. Also, orbit may not be unique. For example, for the character table of elementary abelian group of order 8 there are 28 permutation equivalent matrices and 5 different orbits.

In the next proposition, we see the effect of a (homogeneous) Fourier matrix in a left orbit of symmetric matrices on the remaining elements of the orbit. Though we remark that the following proof also works for right orbit of symmetric matrices.

**Proposition 32.** *Let  $V$  be an orbit of symmetric matrices under the action of  $k$ -cycles.*

- (i) *If  $V$  has a Fourier matrix then every element of the orbit is a Fourier matrix.*
- (ii) *If  $V$  has a homogenous Fourier matrix then every element of the orbit is a homogeneous Fourier matrix.*

*Proof.* For (i). By Proposition 32, each element of  $V$  is a Fourier matrix.

For (ii). Since  $P_{11} = 1$  for each  $P$ , the permutation matrices does not permute the first row and the first column of the elements of an orbit of symmetric matrices. Hence the assertion follows from Part (i).  $\square$

In the next theorem we summarize the conditions under which each element in an orbit of symmetric matrices under the action of 2-cycle forms a modular datum if there is an element of the orbit that forms a modular datum. Although the next theorem seems to have strong conditions, especially part (iv), but it has nice applications, see Example 36.

**Theorem 33.** *Let  $S$  be a non-singular symmetric matrix and  $P$  be a permutation matrix of order 2. Let  $V$  be a left (right) orbit of symmetric matrices under the action of 2-cycle.*

- (i) *If  $S$  is a Fourier matrix and  $P_{11} = 1$  then the structure constants of  $SP$  and  $PS$  exactly match with the corresponding structure constants of  $S$ .*
- (ii) *If  $(S, T)$  is a modular datum then  $(S, PTP)$  is a modular datum.*
- (iii) *If  $(S, T)$  is a modular datum then  $(STP)^2 = S^2$  if and only if  $(SP, PTP)$  is a modular datum,*
- (iv) *Let  $S$  be in  $V$ . Let  $T$  be a diagonal matrix as in Lemma 30 and the multiplication of the permutation matrices with the diagonal matrix  $T$  has the same effect as in Lemma 30 and  $(S, T)$  be a modular datum. Then for each element of  $V$  there is a diagonal matrix that can be determined from modular datum associated with the principal symmetric matrix.*

*Proof.* For (i). Let us consider the two columns  $j$  and  $k$  permuted by the right multiplication of  $P$ . Since  $SP$  is a symmetric matrix, for all  $0 \leq i \leq r-1$ ,  $S_{ij} = S_{ik}$ ,  $S_{jj} = S_{kk}$  and  $S_{jk} \neq S_{jj}$ . Again as  $S$  is a symmetric matrix, therefore, for all  $0 \leq i \leq r-1$ ,  $S_{ji} = S_{ki}$  and  $S_{jk} \neq S_{jj}$ . Hence the structure constants of the matrix  $SP$  exactly matches with the corresponding structure constants of the matrix  $S$ . Similarly, the structure constants of the matrix  $PS$  exactly matches with the corresponding structure constants of the matrix  $S$ .

For (ii). By Proposition 32,  $SP = PS$ . Therefore,  $(ST)^3 = S^2$  implies  $(SPTP)^3 = S^2$ .

For (iii). Since  $(S, T)$  is a modular datum,  $(ST)^3 = S^2$ . By Proposition 32,  $(SP)^2 = S^2$ . The assertion is true as  $(SP, PTP)$  is a modular datum if and only if  $(SPPTP)^3 = (SP)^2$ .

For (iv). The result follows from Lemma 30.  $\square$

In the next lemma, we prove that for a permutation matrix  $P$  of order 3 and modular datum  $(S, T)$  if  $PS$  is symmetric matrix then  $(PS, P^TTP)$  is a modular datum.

**Lemma 34.** *Let  $(S, T)$  be a modular datum. Let  $P$  be a permutation matrix of order 3. Then*

- (i) *If  $PS$  is a symmetric matrix then  $(PS, P^TTP)$  is a modular datum,*
- (ii) *If  $SP$  is a symmetric matrix then  $(SP, PTP^T)$  is a modular datum.*

*Proof.* For (i). Trivially,  $SP$  is a unitary matrix and the set of the structure constants of  $SP$  is equal to the set of the structure constants of  $S$ . Also  $(ST)^3 = S^2$  implies  $(PSP^TTP)^3 = P^2S^2P = (PS)^2$ , see Proposition 35. Hence  $(PS, P^TTP)$  is a modular datum.

For (ii). Similar to the Part (i) above.  $\square$

In the following theorem we prove that for orbits of symmetric matrices under the action of 3-cycles the modular data for each element is determined completely by any element of the orbit. Also we prove that for each rank orbit of homogeneous symmetric matrices is unique.

**Theorem 35.** *Let  $V$  be an orbit of symmetric matrices under the action of 3-cycles. Let an element  $S$  in the orbit  $V$  forms a modular datum  $(S, T)$ . Then each element of  $V$  forms a modular datum and they have equal number of diagonal matrices determined completely by the diagonal matrices for  $S$ .*

*Proof.* The proof follows from Proposition 32 and Lemma 34.  $\square$

The tensor product of two modular data is a modular datum, see [10]. Thus the structure constants of the tensor product of two Fourier matrices are integers. Therefore, by the mixed product property, tensor product of two quasi-modular data is a quasi-modular datum. In the following example we give the applications of the above results to find the  $T$  matrices for the two Fourier matrices of rank 4 given by Cuntz, see [10]. These Fourier matrices are not tensor product of any lower rank Fourier matrices. But all the three matrices are in principal orbit of symmetric matrices that form modular data with equal number (twelve) of diagonal matrices. The next example, also, gives us an impression that the above results can be further extended.

**Example 36.** Consider the Fourier matrix  $S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ . Let  $T = \text{diag}(x, \zeta x)$  be the corresponding diagonal matrix. Since  $(ST)^3 = S^2$ , we have

$$T \in \{ \text{diag}(\zeta_{24}^7, \zeta_{24}^{13}), \text{diag}(\zeta_{24}^{15}, \zeta_{24}^{21}), \text{diag}(\zeta_{24}^{23}, \zeta_{24}^5), \text{diag}(\zeta_{24}, \zeta_{24}^{19}), \text{diag}(\zeta_{24}^9, \zeta_{24}^3), \text{diag}(\zeta_{24}^{17}, \zeta_{24}^{11}) \}.$$

For the corresponding Allen matrix  $s$ , the character table of elementary abelian group of order 4,

$$\tilde{s} = s \otimes s = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

For the corresponding Fourier matrix  $\tilde{S}$ , the diagonal matrix  $\tilde{T}$  belong to the set of the tensor product of the above set  $T$  of matrices. Therefore,

$$\tilde{T} \in \{ \text{diag}(\zeta_{24}^{14}, \zeta_{24}^{20}, \zeta_{24}^{20}, \zeta_{24}^2), \text{diag}(\zeta_{24}^{22}, \zeta_{24}^4, \zeta_{24}^4, \zeta_{24}^{10}), \text{diag}(\zeta_{24}^6, \zeta_{24}^{12}, \zeta_{24}^{12}, \zeta_{24}^{18}), \text{diag}(\zeta_{24}^8, \zeta_{24}^2, \zeta_{24}^{14}, \zeta_{24}^8), \\ \text{diag}(\zeta_{24}^{16}, \zeta_{24}^{10}, \zeta_{24}^{22}, \zeta_{24}^{16}), \text{diag}(1, \zeta_{24}^{18}, \zeta_{24}^6, 1), \text{diag}(\zeta_{24}^8, \zeta_{24}^{14}, \zeta_{24}^2, \zeta_{24}^8), \text{diag}(\zeta_{24}^{16}, \zeta_{24}^{22}, \zeta_{24}^{10}, \zeta_{24}^{16}), \\ \text{diag}(1, \zeta_{24}^6, \zeta_{24}^{18}, 1), \text{diag}(\zeta_{24}^2, \zeta_{24}^{20}, \zeta_{24}^{20}, \zeta_{24}^{14}), \text{diag}(\zeta_{24}^{10}, \zeta_{24}^4, \zeta_{24}^4, \zeta_{24}^{22}), \text{diag}(\zeta_{24}^{18}, \zeta_{24}^{12}, \zeta_{24}^{12}, \zeta_{24}^6) \}.$$

The following matrix  $s_1$  is obtained with the transposition  $(2, 3)$  on the columns (rows) of the matrix  $\tilde{s}$  above. That is,  $s_1$  is the multiplication of the matrix  $\tilde{s}$  with a permutation matrix that permutes the columns (rows) 2 and 3. The corresponding set  $\tilde{T}_1$  of the diagonal matrices is obtained from  $\tilde{T}$  by using Lemma 30 and Part (iv) of Theorem 33.

$$s_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

$$\begin{aligned} \tilde{T}_1 \in & \{ \text{diag}(\zeta_3, \zeta_3, \zeta_3, -\zeta_3), \text{diag}(\zeta_3, \zeta_3, -\zeta_3, \zeta_3), \text{diag}(\zeta_3, -\zeta_3, \zeta_3, \zeta_3), \text{diag}(-\zeta_3, \zeta_3, \zeta_3, \zeta_3), \\ & \text{diag}(\zeta_3^2, \zeta_3^2, \zeta_3^2, \zeta_6), \text{diag}(\zeta_3^2, \zeta_3^2, \zeta_6, \zeta_3^2), \text{diag}(\zeta_3^2, \zeta_6, \zeta_3^2, \zeta_3^2), \text{diag}(\zeta_6, \zeta_3^2, \zeta_3^2, \zeta_3^2), \\ & \text{diag}(1, 1, 1, -1), \text{diag}(1, 1, -1, 1), \text{diag}(1, -1, 1, 1), \text{diag}(-1, 1, 1, 1) \}. \end{aligned}$$

The following matrix  $s_2$  is obtained with the permutation  $(2, 4, 3)$  on the columns (rows) of the matrix  $\tilde{s}$  above. The corresponding set  $\tilde{T}_2$  of the diagonal matrices is obtained from  $\tilde{T}$  by using Lemma 34.

$$s_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}.$$

$$\begin{aligned} \tilde{T}_2 \in & \{ \text{diag}(\zeta_{24}^{14}, \zeta_{24}^2, \zeta_{24}^{20}, \zeta_{24}^{20}), \text{diag}(\zeta_{24}^{22}, \zeta_{24}^{10}, \zeta_{24}^4, \zeta_{24}^4), (\zeta_{24}^6, \zeta_{24}^{18}, \zeta_{24}^{12}, \zeta_{24}^{12}), \text{diag}(\zeta_{24}^8, \zeta_{24}^8, \zeta_{24}^2, \zeta_{24}^{14}), \\ & \text{diag}(\zeta_{24}^{16}, \zeta_{24}^{16}, \zeta_{24}^{10}, \zeta_{24}^{22}), \text{diag}(1, 1, \zeta_{24}^6, \zeta_{24}^{18}), \text{diag}(\zeta_{24}^8, \zeta_{24}^8, \zeta_{24}^{14}, \zeta_{24}^2), \text{diag}(\zeta_{24}^{16}, \zeta_{24}^{16}, \zeta_{24}^{22}, \zeta_{24}^{10}), \\ & \text{diag}(1, 1, \zeta_{24}^{18}, \zeta_{24}^6), \text{diag}(\zeta_{24}^2, \zeta_{24}^{14}, \zeta_{24}^{20}, \zeta_{24}^{20}), \text{diag}(\zeta_{24}^{10}, \zeta_{24}^{22}, \zeta_{24}^4, \zeta_{24}^4), \text{diag}(\zeta_{24}^{18}, \zeta_{24}^6, \zeta_{24}^{12}, \zeta_{24}^{12}) \}. \end{aligned}$$

The matrix  $s_2$  can also be obtained from the matrix  $s_1$  with the permutation  $(3, 4)$  on rows (or columns) of  $s_1$ . Therefore, the corresponding set  $\tilde{T}_2$  of diagonal matrices can also be obtained from the set  $\tilde{T}_1$  by using Lemma 30 and Part (iv) of Theorem 33.

## REFERENCES

- [1] Z. Arad, E. Fisman, and M. Muzychuk, Generalized table algebras, *Israel J. Math.*, **114** (1999), 29-60.
- [2] A. Hanaki and I. Miyamoto, Classification of Small Association Schemes. (<http://math.shinshu-u.ac.jp/~hanaki/as/>)
- [3] Javad Bagherian and Amir Rahnamai Barghi, Standard character condition for C-algebras, arXiv:0803.2423 [math.RT].
- [4] E. Bannai, Association Schemes and Fusion Algebras, *Journal of Algebraic Combinatorics* 2 (1993), 327-344.
- [5] E. Bannai and E. Bannai, Spin Models on Finite Cyclic Groups, *Journal of Algebraic Combinatorics* 3 (1994), 243-259.
- [6] Harvey I. Blau, Quotient structures in C-algebras, *J. Algebra*, **175** (1995), no. 1, 24-64; Erratum: **177** (1995), no. 1, 297-337.
- [7] Harvey I. Blau, Table algebras, *European J. Combin.*, **30** (2009), no. 6, 1426-1455.
- [8] Michael Cuntz, Fusion algebras for imprimitive complex reflection groups, *Journal of Algebra* **311** (2007) 251-267.
- [9] Michael Cuntz, Fusion algebras with negative structure constants, *Journal of Algebra* **319** (2008) 4536-4558.
- [10] Michael Cuntz, Integral modular data and congruences, *J Algebr Comb* **29**, (2009), 357-387.
- [11] Terry Gannon, The automorphisms of affine fusion rings, *Adv. Math.* **165** (2) (2002) 165-193.
- [12] Terry Gannon, Modular data: the algebraic combinatorics of conformal field theory. *J. Algebraic Combin.* **22**(2), 211-250 (2005).

- [13] Allen Herman and Gurmail Singh, Central torsion units of integral reality-based algebras with positive degree map, *International Electronic Journal of Algebra*, accepted.
- [14] Allen Herman and Gurmail Singh, On the Torsion Units of Integral Adjacency Algebras of Finite Association Schemes, *Algebra*, Vol. 2014, 2014, Article ID 842378, 5 pages.
- [15] Allen Herman and Gurmail Singh, Torsion units of integral C-algebras, *JP Journal of Algebra, Number Theory, and Applications*, 36 (2), 2015, 141-155.
- [16] D. G. Higman, Coherent algebras, *Linear Algebra Appl.*, **93** (1987), 209-239.
- [17] A. Hosseini and A. Rahnamai Barghi, Table algebras of rank 3 and its applications to strongly regular graphs, *Journal of Algebra and Its Applications*, **12** (5) (2013), 125-141.
- [18] Gurmail Singh and Allen Herman, Orders of torsion units of integral reality-based algebras with positive degree map and rational multiplicities, submitted.
- [19] Bangteng Xu, On Isomorphisms Between Integral Table Algebras and Applications to Finite Groups and Association Schemes, *Comm. Algebra*, **42** (2014), no. 12, 5249-5263.
- [20] Bangteng Xu, Characters of table algebras and applications to association schemes, *Journal of Combinatorial Theory*, Series A **115** (2008) 1358-1373

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF REGINA, REGINA, CANADA, S4S 0A2  
*E-mail address:* Gurmail.Singh@uregina.ca